Hyperfinite graphings, part I

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Bipartite graphs

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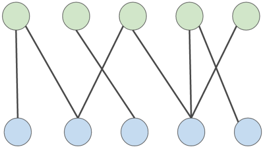
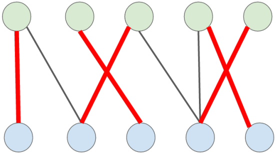
Equivalently, a graph G is bipartite if $V(G)$ **can be partitioned into two sets** such that the edges of G join only vertices in different parts of the partition.



Perfect matchings

A **perfect matching** in a graph G is an involution whose graph is contained in G.



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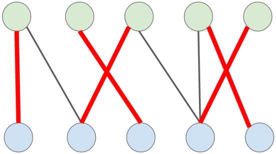


Hall's matching theorem

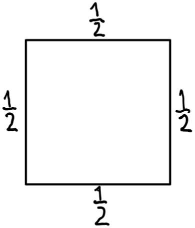
A graph admits a perfect matching if and only if it satisfies the **Hall condition:** $|N_G(A)| \geq |A|$ for every (finite) subset $A \subseteq V(G)$ (here $N_G(A)$ denotes the set of neighbours of A in G).

Fractional perfect matching

A **fractional perfect matching** in a graph G is a function $\varphi : E(G) \to [0,1]$ such that

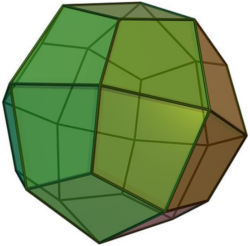
$$\sum_{y \in N_G(x)} \varphi(y) = 1$$

for every $x \in V(G)$.



The Edmonds polytope

Given a bipartite graph G, the set of all **fractional perfect matchings in G forms a convex compact set** (possibly empty).



Theorem (Edmonds)

If a bipartite graph admits a fractional perfect matching, then it admits a perfect matching

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Proof

Any convex compact set has an **extreme point**.

Claim

An **extreme point** φ of the Edmonds polytope of a bipartite graph **is a perfect matching**.

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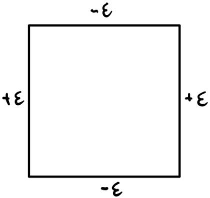
Proof

Suppose φ is not a perfect matching. The set

$$F = \{e \in E(G) : 0 < \varphi(e) < 1\}$$

must contain a cycle.

Note that we can **add** ε **on the even edges** of the cycle and **subtract** ε **on the odd edges**. Write φ_+ for this fractional perfect matching.



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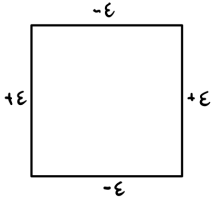
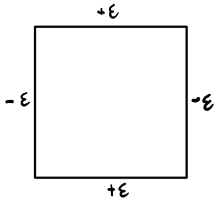


Note that we can also **subtract** ε **on the even edges** of the cycle and **add** ε **on the odd edges**. Write φ_- for this fractional perfect matching.



But now

$$\varphi = \frac{\varphi_{+\varepsilon} + \varphi_{-\varepsilon}}{2},$$

which contradicts that φ was an extreme point.

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This ends the proof of the Claim.

A graph is **locally finite** if the degree of every vertex is finite.

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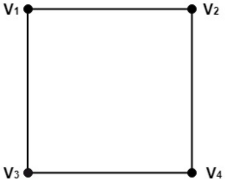
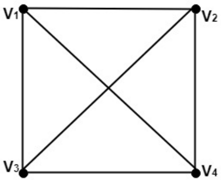
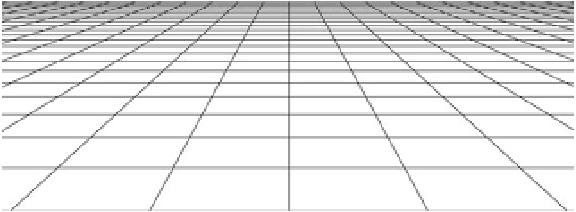
A graph is r-**regular** if the degree of every vertex is equal to r. A graph is **regular** if it is r-regular for some r.





One-ended graphs

An infinite connected locally finite graph is **one-ended** if after removing finitely many vertices, it always has only **one infinite component**.



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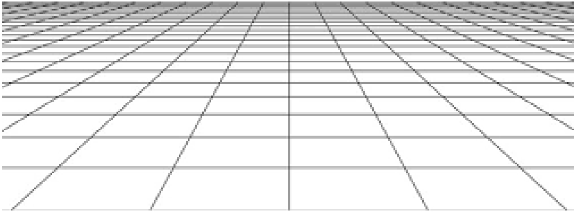

One-ended Cayley graphs

The Cayley graph of $\mathbb{Z}^d$ is one-ended whenever $d > 1$.

Two-ended graphs

An infinite connected graph locally finite is **two-ended** if it is not one-ended and after removing fintiely many vertices it always has at most**two infinite connected components**.

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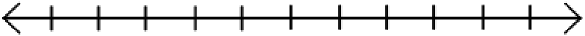
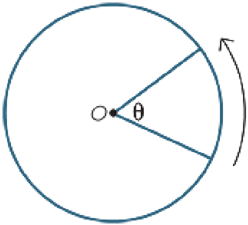


Two-ended Cayley graphs

A Cayley graph of a group Γ is two-ended if and only if Γ is virtually $\mathbb{Z}$.

Example

Let $\theta \in [0,1]$ be irrational and consider the rotation $T_\theta : S^1 \to S^1$ by $2\pi \cdot \theta$. This gives a graph on $V = S^1$ where we put an edge between x and $T_\theta(x)$ for every $x \in S^1$.



Definition

A **Borel graph** is a graph (V, E) where V is a standard Borel space and E is a symmetric Borel subset of $V \times V$.

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Schreier graphs

Suppose Γ is a finitely generated group with a finite symmetric generating set $S \subseteq \Gamma$. If $\Gamma \curvearrowright V$ is a Borel action of Γ on a standardd Borel space V, then we define the **Schreier graph** on V by putting an edge between x and y if $s \cdot x = y$ for one of the generators $s \in S$.

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The previous example is the Schreier graph of the induced action of $\mathbb{Z}$ on the circle.

Probability measure preserving (pmp) graphs

Suppose V is endowed with a Borel probability measure ν. A Borel graph $G = (V, E)$ is **probability measure preserving (pmp)** if for every Borel bijection $f : V \to V$ such that $f \subseteq E$ we have that f preserves ν.

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Graphings

We will refer to Borel pmp graphs as to **graphings**.

Cost

Given a graphing G on (V, ν), there is a natural probability measure on the set of edges $E(G)$ called the **cost**, which we denote by μ.

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It is defined for $F \subseteq E(G)$ as

$$\mu(F) = \mathbf{\frac{1}{2} \int_V \deg_F(x) d}\nu,$$

where $\deg_F(x)$ is the degree in the spanning subgraph induced by F.

Half of the average degree

For a spanning subgraphing F, its cost equals to the **half of the average degree** of F.

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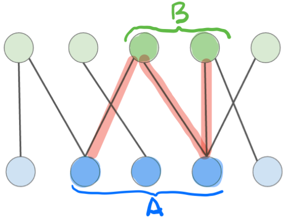
Alternately, (e.g. for bipartite graphs) the cost of a set of edges of the form $A \times B$ is equal to

$$\mu(A \times B) = \int_A \deg_B(x) d\nu(x).$$



Hyperfiniteness

A graphing G is **hyperfinite** if we can write the set of edges

$$E(G) = \bigcup_{n=1}^{\infty} F_n \quad \text{(a.e.)}$$

as an increasing union such that each F_n is a Borel graph and the graph spanned by F_n **has finite connected components**.

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Equivalently, a graphing is hyperfinite if for every $\varepsilon > 0$ there exists $V' \subseteq V$ with $\nu(V \setminus V') < \varepsilon$ suh that G has **finite components on** V'.

Hyperfinite equivalence relations (Slaman–Steel, Weiss)

A graphing G is hyperfinite if and only if the induced equivalence relation (whose classes are the connected components of G) is a **hyperfinite equivalence relation** (a.e.).

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Hyperfinite graphings appear as **Benjamini–Schramm limits** of hyperfinite sequences of finite graphs.

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Property testing

Hyperfinite graphings and sequences are used also in theoretical computer science in **property testing**.

Amenable groups

A group Γ is **amenable** if it admits an invariant finitely additive probability measure on $P(\Gamma)$.

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Schreier graphings of amenable groups

If Γ is amenable, then the Schreier graphing of any Borel action of Γ is hyperfinite

Theorem (Adams)

If a graphing G is hyperfinite, then (a.e.) component of G has **at most** 2 **ends**.

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Examples

The Schreier graphing of any **free action of an one-ended group** is one-ended.

Theorem (Bowen–Kun–S.)

Any bipartite hyperfinite a.e. one-ended regular graphing admits a **measurable perfect matching**.

Aside of the Axiom of Choice

The proof of the theorem can be written without the use of the Axiom of Choice

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However, without going into the details, one we can argue abstractly that the Axiom of Choice is not needed.

First let a be a real coding the Borel graphing and the measure. **The theorem holds in $L[a]$ as $L[a] \models$ ZFC**. Let M be a Borel set in $L[a]$ which is a.e. a perfect matching in $L[a]$

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Absoluteness

The same Borel set is a.e. a perfect matching in V as this is a $\mathbf{\Pi}^1_1$ statement:

$$\forall^\mu x \exists^! y (x, y) \in G \cap M,$$

hence **absolute**.